Resit Exam — Ordinary Differential Equations (WIGDV-07)

Thursday 28 January 2016, 14.00h–17.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 points)

Solve the following differential equation:

$$y' = \frac{y^2 + 2xy}{x^2}, \qquad x > 0$$

Problem 2 (3 + 6 + 6 points)

Consider the differential equation

$$e^{-x} + y + xy + (x + 2e^{-x})\frac{dy}{dx} = 0$$

(a) Show that the equation is *not* exact.

(b) Compute an integrating factor of the form $M(x, y) = \phi(x)$.

(c) Solve the differential equation.

Problem 3 (10 + 5 points)

Consider the following inhomogeneous system:

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 10e^t \end{bmatrix}$$

(a) Compute a *real-valued* fundamental matrix for the homogeneous equation.

(b) Compute a particular solution of the inhomogeneous equation.

Hint: make an educated guess!

Problem 4 (5 + 5 + 5 + 5 points)

Let $a > 0$ be arbitrary, and let $C([0, a])$ denote the space of continuous functions on the interval $[0, a]$. The norm

$$\|y\| = \sup\left\{|y(x)|e^{-3x} \,:\, x \in [0, a]\right\}$$

turns $C([0, a])$ into a Banach space. Consider the integral operator

$$T : C([0, a]) \to C([0, a]), \qquad (Ty)(x) = x + \int_0^x \sin^2(y(t))\, dt.$$

Prove the following statements:

(a) $\left|\sin^2(y) - \sin^2(z)\right| \leq 2\left|y - z\right| \qquad \forall\, y, z \in \mathbb{R}.$

(b) $\left|(Ty)(x) - (Tz)(x)\right| \leq \dfrac{2(e^{3x} - 1)}{3}\|y - z\| \qquad \forall\, y, z \in C([0, a]),\ x \in [0, a]$

(c) $\|Ty - Tz\| \leq \dfrac{2}{3}\|y - z\| \qquad \forall\, y, z \in C([0, a])$

(d) The initial value problem $y' = 1 + \sin^2(y)$ with $y(0) = 0$ has a unique solution on the interval $[0, a]$.

Problem 5 (3 + 12 points)

Consider the second-order equation:

$$(2x + 1)u'' - 4(x + 1)u' + 4u = 0$$

(a) Compute the value of λ for which $u_1(x) = e^{\lambda x}$ is a solution.

(b) Compute a second solution of the form $u_2(x) = c(x)u_1(x)$ such that u_1 and u_2 are linearly independent.

Problem 6 (15 points)

Compute *all* eigenvalues λ and corresponding eigenfunctions u of the following boundary value problem:

$$u'' + \lambda u = 0, \qquad u(0) - u'(0) = 0, \qquad u(\pi) - u'(\pi) = 0$$

End of test (90 points)

Solution of Problem 1 (10 points)

Solution 1. We can rewrite the equation as

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{2y}{x}$$

Taking the substitution $u = y/x$ gives the new differential equation

$$\frac{du}{dx} = \frac{u^2 + u}{x}$$

(2 points)

Separating the variables gives:

$$\int \frac{1}{u(u+1)}\,du = \int \frac{1}{x}\,dx \quad \Rightarrow \quad \int \frac{1}{u} - \frac{1}{u+1}\,du = \int \frac{1}{x}\,dx$$

(2 points)

Computing the integrals gives

$$\log|u| - \log|1+u| = \log|x| + C \quad \Rightarrow \quad \log\left|\frac{u}{1+u}\right| = \log|x| + C = \log(x) + C$$

where we have used the assumption $x > 0$.
(4 points)

Solving for u and then for y gives

$$\frac{u}{1+u} = Kx \quad \Rightarrow \quad u = \frac{Kx}{1-Kx} \quad \Rightarrow \quad y = \frac{Kx^2}{1-Kx}$$

where $K = \pm e^C$ is an arbitrary constant.
(2 points)

Solution 2. We can rewrite the equation as

$$\frac{dy}{dx} = \frac{2}{x} \cdot y + \frac{1}{x^2} \cdot y^2$$

in which we recognize a Bernoulli equation with $\alpha = 2$. Therefore, we take the substitution $z = y^{1-\alpha} = y^{-1}$.
(1 point)

The new variable satisfies a linear differential equation:

$$z' + \frac{2}{x} \cdot z = -\frac{1}{x^2}$$

(3 points)

We can solve this equation by multiplying with the integrating factor x^2:

$$x^2 z' + 2xz = -1 \quad \Rightarrow \quad \left[x^2 z\right]' = -1 \quad \Rightarrow \quad x^2 z = -x + C \quad \Rightarrow \quad z = \frac{C - x}{x^2}$$

(5 points)

Finally, we obtain

$$y = \frac{1}{z} = \frac{x^2}{C - x}$$

(1 point)

This is equivalent with the previous solution via the relation $C = 1/K$.

Solution of Problem 2 (3 + 6 + 6 points)

(a) Define the functions

$$g(x,y) = e^{-x} + y + xy \quad \text{and} \quad h(x,y) = x + 2e^{-x}$$

Then $g_y = 1 + x$ and $h_x = 1 - 2e^{-x}$. Since $g_y \neq h_x$ the equation is not exact.
(3 points)

(b) The function $M(x,y) = \phi(x)$ is an integrating factor if and only if

$$\frac{\partial}{\partial y}\big[\phi(x)(e^{-x} + y + xy)\big] = \frac{\partial}{\partial x}\big[\phi(x)(x + 2e^{-x})\big]$$

Expanding the derivatives gives

$$\phi(x)(1 + x) = \phi'(x)(x + 2e^{-x}) + \phi(x)(1 - 2e^{-x})$$

or, equivalently,

$$\phi'(x) = \phi(x)$$

(5 points)

Therefore, an integrating factor is given by $M(x,y) = e^x$.
(1 point)

(c) After multiplication with the integrating factor the equation reads as

$$1 + ye^x + xye^x + (xe^x + 2)\frac{dy}{dx} = 0$$

Define a potential function by

$$F(x,y) = \int xe^x + 2\,dy = xye^x + 2y + C(x)$$

(3 points)

This function should also satisfy

$$F_x = 1 + ye^x + xye^x \quad \Rightarrow \quad ye^x + xye^x + C'(x) = 1 + ye^x + xye^x$$

From this it follows that $C'(x) = 1$ so we can take $C(x) = x$.
(2 points)

Finally, the solution in implicit form reads as

$$xye^x + 2y + x = K$$

where K is an arbitrary constant.
(1 point)

Solution of Problem 3 (10 + 5 points)

(a) First, compute the characteristic polynomial of the coefficient matrix:

$$\det\begin{bmatrix} -\lambda & 1 \\ -5 & -4-\lambda \end{bmatrix} = \lambda^2 + 4\lambda + 5$$

Hence, the eigenvalues of the coefficient matrix are $\lambda = -2 \pm i$.
(3 points)

For $\lambda = -2 + i$ an associated eigenvector is given by $\mathbf{v} = \begin{bmatrix} 1 & -2+i \end{bmatrix}^\top$. Since the matrix is real an associated eigenvector for $\lambda = -2 - i$ is given by $\bar{\mathbf{v}}$.
(3 points)

We can write two linearly independent solutions of the homogeneous equation:

$$\mathbf{y}_1 = \mathrm{Re}(e^{(-2+i)t}\mathbf{v}) = \begin{bmatrix} e^{-2t}\cos(t) \\ e^{-2t}(-2\cos(t) - \sin(t)) \end{bmatrix}$$

$$\mathbf{y}_2 = \mathrm{Im}(e^{(-2+i)t}\mathbf{v}) = \begin{bmatrix} e^{-2t}\sin(t) \\ e^{-2t}(-2\sin(t) + \cos(t)) \end{bmatrix}$$

(3 points)

This gives the following fundamental matrix:

$$Y(t) = e^{-2t}\begin{bmatrix} \cos(t) & \sin(t) \\ -2\cos(t) - \sin(t) & -2\sin(t) + \cos(t) \end{bmatrix}$$

(1 point)

(b) We use the following educated guess:

$$\mathbf{y}_p = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$$

where A and B are undetermined coefficients. Substitution in the differential equation gives

$$\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} = \begin{bmatrix} Be^t \\ (-4A - 5B + 10)e^t \end{bmatrix}$$

for which the only solution is $A = B = 1$.
(5 points)

Solution of Problem 4 (5 + 5 + 5 + 5 points)

(a) The Mean Value Theorem implies that for all $y, z \in \mathbb{R}$ there exists a point t between y and z such that

$$\sin^2(y) - \sin^2(z) = 2\sin(t)\cos(t)(y - z)$$

(3 points)

Taking absolute values on each side gives the desired result.
(2 points)

(b) Let $y, z \in C([0, a])$ be arbitrary. Using the triangle inequality and part (a) gives

$$\begin{aligned} \big|(Ty)(x) - (Tz)(x)\big| &= \left|\int_0^x \sin^2(y(t)) - \sin^2(z(t))\,dt\right| \\ &\le \int_0^x \big|\sin^2(y(t)) - \sin^2(z(t))\big|\,dt \\ &\le \int_0^x 2\big|y(t) - z(t)\big|\,dt \end{aligned}$$

(3 points)

Noting that $|y(t) - z(t)|e^{-3t} \le \|y - z\|$ for all $t \in [0, a]$ gives

$$\begin{aligned} \big|(Ty)(x) - (Tz)(x)\big| &\le \int_0^x 2\big|y(t) - z(t)\big|e^{-3t}e^{3t}\,dt \\ &\le 2\|y - z\| \int_0^x e^{3t}\,dt \\ &= \frac{2(e^{3x} - 1)}{3}\|y - z\| \end{aligned}$$

(2 points)

(c) From part (b) it follows that

$$\big|(Ty)(x) - (Tz)(x)\big|e^{-3x} \le \frac{2(1 - e^{-3x})}{3}\|y - z\| \le \frac{2}{3}\|y - z\|$$

Taking the supremum over the interval $[0, a]$ gives the desired inequality.
(5 points)

(d) Note that we have the following equivalences:

$$Ty = y \quad \Leftrightarrow \quad y(x) = x + \int_0^x \sin^2(y(t))\,dt \quad \Leftrightarrow \quad \begin{cases} y' &= 1 + \sin^2(y) \\ y(0) &= 0 \end{cases}$$

From part (c) it follows that the operator $T : C([0, a]) \to C([0, a])$ is a contraction. Therefore, Banach's fixed point theorem implies the existence of a unique $y \in C([0, a])$ such that $Ty = y$, which implies the existence and uniqueness result for the initial value problem.
(5 points)

Solution of Problem 5 (3 + 12 points)

Consider the second-order equation:

$$(2x+1)u'' - 4(x+1)u' + 4u = 0$$

(a) We have

$$u_1 = e^{\lambda x} \quad \Rightarrow \quad u_1' = \lambda e^{\lambda x} \quad \Rightarrow \quad u_1'' = \lambda^2 e^{\lambda x}$$

Substitution in the differential equation gives

$$\lambda^2(2x+1)e^{\lambda x} - 4\lambda(x+1)e^{\lambda x} + 4e^{\lambda x} = 0 \quad \Rightarrow \quad 2\lambda(\lambda-2)x + (\lambda-2)^2 = 0$$

The only solution is $\lambda = 2$ which gives $u_1(x) = e^{2x}$.
(3 points)

(b) We have

$$u_2 = c(x)e^{2x} \quad \Rightarrow \quad u_2' = \big[c'(x) + 2c(x)\big]e^{2x}$$
$$\Rightarrow \quad u_2'' = \big[c''(x) + 4c'(x) + 4c(x)\big]e^{2x}$$

Substitution in the differential equation shows that u_2 is a solution if and only if the function $c(x)$ satisfies the following equation:

$$(2x+1)c''(x) + 4xc'(x) = 0$$

(3 points)

Setting $z(x) = c'(x)$ gives an equation with separated variables:

$$\frac{dz}{dx} = -\frac{4x}{2x+1}z = \left(-2 + \frac{2}{2x+1}\right)z$$

(2 points)

The solution is given by

$$z(x) = K\exp\left(-2x + \log|2x+1|\right) = K(2x+1)e^{-2x}$$

(2 points)

Finally, integration by parts gives

$$c(x) = \int K(2x+1)e^{-2x} = -K(x+1)e^{-2x} + C$$

An obvious choice for the constants is $K = -1$ and $C = 0$ which gives $u_2(x) = x + 1$ as a second solution.
(2 points)

Clearly, u_1 and u_2 are linearly independent since an exponential function is not a constant multiple of a linear function.
(1 point)

Solution of Problem 6 (15 points)

The case $\lambda = 0$. This gives the general solution $u = c_1 x + c_2$. The boundary conditions imply that

$$\begin{bmatrix} -1 & 1 \\ \pi - 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the coefficient matrix has nonzero determinant we conclude that $c_1 = c_2 = 0$ which only gives the trivial solution $u = 0$. Therefore, $\lambda = 0$ is not an eigenvalue.
(3 points)

The case $\lambda < 0$. If we write $\lambda = -\mu^2$ then we get the general solution

$$u = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

The boundary conditions imply that

$$\begin{bmatrix} 1 - \mu & 1 + \mu \\ (1 - \mu)e^{\mu\pi} & (1 + \mu)e^{-\mu\pi} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(2 points)

For the existence of nontrivial solutions we need the determinant of the coefficient matrix to be zero:

$$(1 - \mu^2)(e^{-\mu\pi} - e^{\mu\pi}) = 0 \quad \Leftrightarrow \quad \mu = \pm 1$$

Taking $\mu = 1$ gives the eigenvalue $\lambda_0 = -1$ and the corresponding eigenfunction $u_0 = e^x$.
(4 points)

The case $\lambda > 0$. If we write $\lambda = \mu^2$ then we get the general solution

$$u = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

The boundary conditions imply that

$$\begin{bmatrix} 1 & -\mu \\ \cos(\mu\pi) + \mu \sin(\mu\pi) & \sin(\mu\pi) - \mu \cos(\mu\pi) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(2 points)

For the existence of nontrivial solutions we need the determinant of the coefficient matrix to be zero:

$$(1 + \mu^2)\sin(\mu\pi) = 0 \quad \Leftrightarrow \quad \mu \in \mathbb{Z}$$

Taking $\mu = n \in \mathbb{N}$ gives the eigenvalues $\lambda_n = n^2$ and the corresponding eigenfunctions $u_n = n\cos(nx) + \sin(nx)$.
(4 points)